Math 3450 - Homework # 1 - Part B - Solutions

Part 1 - Conceptual

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1. True or False: $\{1\} \in \mathcal{P}(\{1,2\})$

   Solution: True. $\{1\} \in \mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

2. True or False: $\{1\} \subseteq \mathcal{P}(\{1,2\})$

   Solution: False. $\{1\} \subseteq \mathcal{P}(\{1,2\})$ would mean that $1 \in \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ which is not true.

3. Show that the following if-then statement is false by giving a counter-example: Let A, B, C be sets. If $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$, then $A \cap C \neq \emptyset$.

   Solution: Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{3,4\}$. Then $A \cap B = \{2\} \neq \emptyset$ and $B \cap C = \{3\} \neq \emptyset$, however $A \cap C = \emptyset$.

Part 2 - Proofs

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4. Prove that $\{12n \,|\, n \in \mathbb{Z}\} \subseteq \{2n \,|\, n \in \mathbb{Z}\} \cap \{3n \,|\, n \in \mathbb{Z}\}$.

   Proof. Let $x \in \{12n \,|\, n \in \mathbb{Z}\}$.

   Then $x = 12n$ for some $n \in \mathbb{Z}$.

   Note that $x = 2(6n) = 2k$ where $k = 6n$ is an integer (because the set of integers is closed under multiplication).

   Thus $x \in \{2n \,|\, n \in \mathbb{Z}\}$.

   Since $x = 3(4n) = 3l$ where $l = 4n$ is an integer (because the set of integers is closed under multiplication).

   Thus $x \in \{3n \,|\, n \in \mathbb{Z}\}$.

   We see then that $x \in \{2n \,|\, n \in \mathbb{Z}\} \cap \{3n \,|\, n \in \mathbb{Z}\}$.

   Therefore $\{12n \,|\, n \in \mathbb{Z}\} \subseteq \{2n \,|\, n \in \mathbb{Z}\} \cap \{3n \,|\, n \in \mathbb{Z}\}$.

   □

5. Prove that $\{9^n \mid n \in \mathbb{Z}\} \subseteq \{3^n \mid n \in \mathbb{Z}\}$, but $\{9^n \mid n \in \mathbb{Z}\} \neq \{3^n \mid n \in \mathbb{Z}\}$.

Proof. Let $x \in \{9^n \mid n \in \mathbb{Z}\}$.

Then $x = 9^n$ where $n \in \mathbb{Z}$.

So $x = 3^{2n} = 3^k$ where $k = 2n$ is an integer.

Thus $x \in \{3^n \mid n \in \mathbb{Z}\}$.

Hence $\{9^n \mid n \in \mathbb{Z}\} \subseteq \{3^n \mid n \in \mathbb{Z}\}$.

Note that $3 \in \{3^n \mid n \in \mathbb{Z}\}$, but $3 \notin \{9^n \mid n \in \mathbb{Z}\}$.

Thus $\{9^n \mid n \in \mathbb{Z}\} \neq \{3^n \mid n \in \mathbb{Z}\}$

□

6. Let $A = \{2k \mid k \in \mathbb{Z}\}$ and $B = \{3n \mid n \in \mathbb{Z}\}$. Prove that $A \cap B = \{6m \mid m \in \mathbb{Z}\}$.

Proof. **(⊆)**

First we show that $A \cap B \subseteq \{6m \mid m \in \mathbb{Z}\}$.

Suppose that $x \in A \cap B$.

Then $x \in A$ and $x \in B$.

Then $x = 2k$ and $x = 3n$ where $k, n \in \mathbb{Z}$.

Thus $2k = 3n$.

Therefore, $3n$ is even.

Since an odd integer multiplied by and odd integer is odd, we cannot have that n is odd.

Therefore n is even.

So $n = 2l$ where $l \in \mathbb{Z}$.

Thus $x = 3n = 3(2l) = 6l \in \{6m \mid m \in \mathbb{Z}\}$.

So $A \cap B \subseteq \{6m \mid m \in \mathbb{Z}\}$.

(⊇)

Now we show that $\{6m \mid m \in \mathbb{Z}\} \subseteq A \cap B$.

Let $x \in \{6m \mid m \in \mathbb{Z}\}$.

Then $x = 6m$ where $m \in \mathbb{Z}$.

Note that $x = 6m = 2(3m) = 3(2m)$.

Hence $x \in A$ and $x \in B$.

Thus $x \in A \cap B$.

So $\{6m \,|\, m \in \mathbb{Z}\} \subseteq A \cap B$.

Therefore by **(⊆)** and **(⊇)** we get that $A \cap B = \{6m \,|\, m \in \mathbb{Z}\}$.

□

7. Let A, B, C, D be sets.

   (a) Prove that if $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

   Proof. Suppose $x \in A \cup C$.
   We will show that $x \in B \cup C$.
   We know that $x \in A$ or $x \in C$.
   Case 1: Suppose that $x \in A$.
   Since $A \subseteq B$ we have that $x \in B$.
   So $x \in B \cup C$.
   Case 2: Suppose that $x \in C$.
   Then $x \in B \cup C$.
   In either case above, we get that $x \in B \cup C$.
   So $A \cup C \subseteq B \cup C$. □

   (b) Prove that if $A \subseteq B$ then $A \subseteq B \cup C$.

   Proof. Suppose that $A \subseteq B$.
   We use this to show that $A \subseteq B \cup C$.
   Let $x \in A$.
   Since $A \subseteq B$ and $x \in A$, we know that $x \in B$.
   Since $x \in B$, we know that $x \in B \cup C$.
   Therefore, if $x \in A$, then $x \in B \cup C$ is true.
   So $A \subseteq B \cup C$. □

   (c) Prove that if $A \subseteq B$, then $A - C \subseteq B - C$.

Proof. Let $x \in A - C$.

We will show that $x \in B - C$.

We know that $x \in A$ and $x \notin C$, because $x \in A - C$.

Since $x \in A$ and $A \subseteq B$ we have that $x \in B$.

Since $x \in B$ and $x \notin C$ it follows that $x \in B - C$.

Therefore $A - C \subseteq B - C$. □

(d) Prove that $A \subseteq B$ if and only if $A - B = \emptyset$.

Proof 1 - by contraposition. In this version of the proof we will use contraposition. Recall that P iff Q is equivalent to $\neg P$ iff $\neg Q$. Thus "$A \subseteq B$ if and only if $A - B = \emptyset$" is equivalent to "$A \not\subseteq B$ if and only if $A - B \neq \emptyset$". We instead prove this second statement.

(⇒) Suppose that $A \not\subseteq B$.

This means that there exists an $x \in A$ with $x \notin B$.

Thus there exists x with $x \in A - B$.

So $A - B \neq \emptyset$.

(⇐) Suppose that $A - B \neq \emptyset$.

Then there exists $x \in A - B$.

So $x \in A$ and $x \notin B$.

Thus $A \not\subseteq B$.

□

Proof 2 - by contradiction. **(⇒)**

First, we will show that if $A \subseteq B$, then $A - B = \emptyset$.

We will prove this by contradiction.

Suppose that $A \subseteq B$, but $A - B \neq \emptyset$.

Then there exists $x \in A - B$.

So $x \in A$ and $x \notin B$.

But $A \subseteq B$, so $x \in A$ implies that $x \in B$.

Contradiction.

Therefore $A - B = \emptyset$.

(⇐)

Next, we will show that if $A - B = \emptyset$, then $A \subseteq B$.

Suppose $x \in A$. We will show that $x \in B$.
Suppose to the contrary that $x \notin B$.
Then $x \in A - B$, since $x \in A$ and $x \notin B$.
But $A - B = \emptyset$.
Contradiction.
Therefore $x \in B$.
Therefore $A \subseteq B$. □

(e) Prove that $A \subseteq B$ if and only if $A \cap B = A$.

Proof. **(⇒)** We first show that if $A \subseteq B$ then $A \cap B = A$.
Suppose that $A \subseteq B$.
We will show that $A \cap B = A$.
We know that $A \cap B \subseteq A$ by the definition of intersection.
Why is $A \subseteq A \cap B$?
Let $x \in A$.
Then $x \in B$ because $A \subseteq B$.
Thus $x \in A$ and $x \in B$.
So $x \in A \cap B$.
Thus $A \subseteq A \cap B$
Since $A \cap B \subseteq A$ and $A \subseteq A \cap B$ we know that $A \cap B = A$.
(⇐)
We now show that if $A \cap B = A$ then $A \subseteq B$.
Suppose that $A \cap B = A$.
We will show that $A \subseteq B$.
Let $x \in A$.
Then $x \in A \cap B$ since $A = A \cap B$.
Thus $x \in B$ since $x \in A \cap B$.
Therefore $A \subseteq B$.

By **(⇒)** and **(⇐)** we have shown that $A \subseteq B$ if and only if $A \cap B = A$.

□

(f) Prove that if $B \subseteq C$, then $A \times B \subseteq A \times C$.

Proof. Suppose that $B \subseteq C$.
Let $x \in A \times B$.
Then $x = (a, b)$ where $a \in A$ and $b \in B$.
Since $B \subseteq C$ and $b \in B$ we know that $b \in C$.
Thus $x = (a, b)$ where $a \in A$ and $b \in C$.
Hence $x \in A \times C$.
Therefore $A \times B \subseteq A \times C$.

□

(g) Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof. **(⊆)**
First, we will show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.
Suppose that $(x, y) \in A \times (B \cap C)$.
Then $x \in A$ and $y \in B \cap C$.
Since $y \in B \cap C$, we have that $y \in B$ and $y \in C$.
Since $x \in A$ and $y \in B$, we have that $(x, y) \in A \times B$.
Since $x \in A$ and $y \in C$, we have that $(x, y) \in A \times C$.
So $(x, y) \in (A \times B) \cap (A \times C)$.
Therefore $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.
(⊇)
Next, we will show that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.
Suppose that $(x, y) \in (A \times B) \cap (A \times C)$.
Then $(x, y) \in A \times B$ and $(x, y) \in A \times C$.
Since $(x, y) \in A \times B$ we get that $x \in A$ and $y \in B$.
Since $(x, y) \in A \times C$ we get that $x \in A$ and $y \in C$.
So $y \in B \cap C$, because $y \in B$ and $y \in C$.
Thus $(x, y) \in A \times (B \cap C)$, because $x \in A$ and $y \in B \cap C$.
Ergo, $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

Therefore by **(⊆)** and **(⊇)** we get that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. □

(h) Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof. **(⊆)** First, we will show that $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$.
Suppose $(x, y) \in (A \times B) \cap (C \times D)$.
Then $(x, y) \in (A \times B)$ and $(x, y) \in (C \times D)$.
So $x \in A$ and $y \in B$, because $(x, y) \in (A \times B)$.
Also, $x \in C$ and $y \in D$, because $(x, y) \in (C \times D)$.
So $x \in A \cap C$, because $x \in A$ and $x \in C$.
Also $y \in B \cap D$, because $y \in B$ and $y \in D$.
So $(x, y) \in (A \cap C) \times (B \cap D)$, because $x \in A \cap C$ and $y \in B \cap D$.
Therefore $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$.
(⊇) Next, we will show that $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$.
Suppose that $(x, y) \in (A \cap C) \times (B \cap D)$.
Then $x \in A \cap C$ and $y \in B \cap D$.
So $x \in A$ and $x \in C$, because $x \in A \cap C$.
Also $y \in B$ and $y \in D$, because $y \in B \cap D$.
So $(x, y) \in A \times B$, because $x \in A$ and $y \in B$.
Also, $(x, y) \in C \times D$, because $x \in C$ and $y \in D$.
Therefore $(x, y) \in (A \times B) \cap (C \times D)$, because $(x, y) \in A \times B$ and $(x, y) \in C \times D$.
So $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$.

Therefore, by **(⊆)** and **(⊇)** we get that$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. □

(i) Prove that $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof. **(⊆)** First, we will show that $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.
Suppose $x \in A \cap (B \cap C)$.
Then $x \in A$ and $x \in B \cap C$.
So $x \in A$ and $x \in B$ and $x \in C$.
Since $x \in A$ and $x \in B$ we have that $x \in A \cap B$.
So $x \in (A \cap B) \cap C$, because $x \in A \cap B$ and $x \in C$.

Therefore, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

($\supseteq$) Now we will show that $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

Let $x \in (A \cap B) \cap C$.

Then $x \in (A \cap B)$ and $x \in C$.

Thus $x \in A$ and $x \in B$ and $x \in C$.

Since $x \in B$ and $x \in C$ we have that $x \in B \cap C$.

Hence $x \in A \cap (B \cap C)$ since $x \in A$ and $x \in B \cap C$.

Therefore, by **($\subseteq$)** and **($\supseteq$)** we get that $A \cap (B \cap C) = (A \cap B) \cap C$. □

(j) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. **($\subseteq$)** First, we will show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Let $x \in A \cup (B \cap C)$.

We know $x \in A$ or $x \in B \cap C$.

<u>Case 1</u>: Suppose that $x \in A$.

Then $x \in A \cup B$, since $x \in A$.

Also, $x \in A \cup C$, since $x \in A$.

Thus $x \in A \cup B$ and $x \in A \cup C$.

So, $x \in (A \cup B) \cap (A \cup C)$.

<u>Case 2</u>: Suppose that $x \in B \cap C$.

Then $x \in B$ and $x \in C$.

So $x \in A \cup B$, because $x \in B$.

Also $x \in A \cup C$, because $x \in C$.

Thus $x \in A \cup B$ and $x \in A \cup C$.

So $x \in (A \cup B) \cap (A \cup C)$.

In either case, we have $x \in (A \cup B) \cap (A \cup C)$.

So $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

($\supseteq$) Next, we will show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Suppose that $x \in (A \cup B) \cap (A \cup C)$.

Then $x \in (A \cup B)$ and $x \in (A \cup C)$.

So $x \in A$ or $x \in B$, because $x \in (A \cup B)$.
Case 1: Suppose that $x \in A$.
Then $x \in A \cup (B \cap C)$, because $x \in A$.
Case 2: Suppose that $x \in B$.
We know that $x \in A$ or $x \in C$, because $x \in (A \cup C)$ (from above before case 1).
We break case 2 into two sub-cases.
Case 2i: Suppose that $x \in A$.
Then $x \in A \cup (B \cap C)$, because $x \in A$.
Case 2ii: Suppose that $x \in C$.
Then $x \in B \cap C$, because $x \in B$ and $x \in C$.
So $x \in A \cup (B \cap C)$, because $x \in B \cap C$.

In every case, we have $x \in A \cup (B \cap C)$.
Therefore $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Therefore, by **(⊆)** and **(⊇)** we get that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ □

8. Let A and B be sets.

   (a) Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

   Proof. **(⊆)** First, we will show that $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.
   Suppose that $S \in \mathcal{P}(A \cap B)$. We will show that $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.
   We know that $S \subseteq A \cap B$, because $S \in \mathcal{P}(A \cap B)$.
   So every element of S is in $A \cap B$.
   So every element of S is in both A and B.
   So $S \subseteq A$ and $S \subseteq B$.
   So $S \in \mathcal{P}(A)$ and $\mathcal{P}(B)$.
   So $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.
   Therefore $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.
   (⊇) Next, we will show that $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.
   Suppose that $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$. We will show that $S \in \mathcal{P}(A \cap B)$.

We know that $S \in \mathcal{P}(A)$ and $\mathcal{P}(B)$, because $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.
So $S \subseteq A$ and $S \subseteq B$.
So every element of S is in both A and B.
So every element of S is in $A \cap B$.
So $S \subseteq A \cap B$.
So $S \in \mathcal{P}(A \cap B)$.
Therefore $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Therefore, by **($\subseteq$)** and **($\supseteq$)** we get that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. □

(b) Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Proof. Suppose that $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$.
Then $S \in \mathcal{P}(A)$ or $S \in \mathcal{P}(B)$.
Case 1: Suppose that $S \in \mathcal{P}(A)$.
Then $S \subseteq A$.
So $S \subseteq A \cup B$, by problem 7b above.
Case 2: $S \in \mathcal{P}(B)$
Then $S \subseteq B$.
So $S \subseteq A \cup B$, by problem 7b above.
In either case, we have $S \subseteq A \cup B$.
So $S \in \mathcal{P}(A \cup B)$.
Thus, if $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$, then $S \in \mathcal{P}(A \cup B)$.
Therefore $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. □

(c) Give an example where $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

Solution:
Let $A = \{1\}, B = \{2\}$.
Then $\mathcal{P}(A) = \{\emptyset, \{1\}\}$.
And $\mathcal{P}(B) = \{\emptyset, \{2\}\}$.
So $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\}$.
Also $A \cup B = \{1, 2\}$.
So $\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
This example satisfies $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

9. Let A and B be sets. Prove that $A - B$ and B are disjoint.

   Proof. We will show that $(A - B) \cap B = \emptyset$.

   We do this by contradiction.

   Suppose that $(A - B) \cap B \neq \emptyset$.

   Then there exists $x \in (A - B) \cap B$.

   So $x \in A - B$ and $x \in B$.

   But $x \in A - B$ implies that $x \in A$ and $x \notin B$.

   Thus we have that $x \in B$ and $x \notin B$.

   Contradiction. (We cannot have both $x \in B$ and $x \notin B$.)

   Therefore $(A - B) \cap B = \emptyset$.

   Therefore $A - B$ and B are disjoint. □

10. Let A and B be sets. Suppose that $B \neq \emptyset$ and $A \times B \subseteq B \times C$. Prove that $A \subseteq C$.

    Proof. Suppose that $B \neq \emptyset$ and $A \times B \subseteq B \times C$.

    Let $a \in A$.

    Since B is not empty there exists some $b \in B$.

    Then $(a, b) \in A \times B$.

    Since $A \times B \subseteq B \times C$ and $(a, b) \in A \times B$, we get that $(a, b) \in B \times C$.

    Thus $a \in B$ and $b \in C$.

    Then $(a, a) \in A \times B$ because $a \in A$ and $a \in B$.

    Again since $A \times B \subseteq B \times C$ and $(a, a) \in A \times B$ we get that $(a, a) \in B \times C$.

    Therefore $a \in C$.

    Hence $A \subseteq C$.

    □